On spectral representations of tensor random fields on the sphere*

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Abstract

We study the representations of tensor random fields on the sphere basing on the theory of representations of the rotation group. Introducing specific components of a tensor field and imposing the conditions of weak isotropy and mean square continuity, we derive their spectral decompositions in terms of generalized spherical functions. The properties of random coefficients of the decompositions are characterized, including such an important question as conditions of Gaussianity.
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1 Introduction

This paper was inspired and motivated by recent advances in the study of cosmic microwave background (CMB) radiation, which is currently a crucial topic of investigation in cosmology. Amazingly, the CMB anisotropies contain a wealth of important cosmological information and provide the main testing ground for the theories of early universe, with the possibility of precise determination of fundamental cosmological parameters. Note that this area has raised a number of issues, theoretical and practical, both in physics and mathematics. Nowadays, with the flood of high quality data available from several satellite missions and expected from the future ones, the focus of CMB research is on developing theoretical tools for data interpretation and analysis.

The CMB radiation is characterized by four Stokes parameters [9]: the intensity I (or, equivalently, the temperature T) and the other three parameters Q, U and V, which define the polarization state, namely, linear and circular polarization (however, the fourth parameter V describing the circular polarization is not necessary in the standard cosmological models).

Much more attention has been paid to the study of temperature fluctuations in cosmic microwave background, great deal of experimental activity has been accompanied by thorough numerical and analytic work. In particular, rigorous mathematical basis has been provided through the excellent series of papers [1], [2], [16]–[21].

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From the mathematical point of view, the temperature fluctuations are considered as (a realization of) a random field on the unit sphere. One natural tool for statistical analysis of random fields is a spectral decomposition, which for scalar random fields on the sphere is a decomposition in the series of spherical harmonics (see, e.g., [13], [32]). These decompositions have been known and used in different applied areas for quite a long time, however recent contributions in [1], [2], [16]–[21] shed more light on many important practical issues such as characterization of spherical harmonics coefficients and their asymptotic behavior, characterization of spectrum and bispectrum, testing for Gaussianity on spherical random fields and some other. Higher order spectra for spherical random fields have been analyzed in [20].

It has been long recognized that polarization in CMB provides a source of information on cosmological parameters complementary to temperature fluctuations. This significant additional information is necessary to create a complete picture of anisotropies used to constrain cosmological models, to improve the accuracy in determining cosmological parameters, and also to study relic gravitational waves, which is one of the biggest challenges today (see, e.g., [34], [3]). However, polarization anisotropies in CMB are detected at much smaller scales than temperature fluctuations and this limitation has been overcome only recently with improved sensitivity and resolution of CMB experiments. First polarization measurements were obtained by WMAP experiment (see, e.g., [25], http://map.gsfc.nasa.gov/) and further data are expected from Plank surveyor satellite (see http://www.sciops.esa.int/) prompting many groups of scientists to focus on polarization research. Last few years of intensive studies in this rapidly expanding area resulted in developing physical and mathematical formalism to describe, characterize and analyze the polarization counterpart in CMB.

From the mathematical point of view, considering polarization we deal again with random fields on the sphere, however, polarization Stokes parameters Q and U are not scalars but rather the components of a symmetric trace-free rank 2 tensor, or, equivalently, the combinations $Q \pm iU$ are spin ± 2 quantities. Therefore, for their mathematical description one needs to consider tensor-valued random fields (or random fields with spin-weighted quantities) on the sphere and their spectral decomposition and statistical analysis.

In order to study polarization anisotropies, analytic methods have been formulated in physical literature based on expansions of tensor-valued functions on the sphere in the series of appropriate tensor spherical harmonics, or expansions in spin-weighted spherical harmonics (this was done initially by two group of scientists in [12] and [33] respectively, and numerous further developments have been elaborated by now). The main reference for the mathematical formalism used in these studies comes from non-probabilistic literature, in particular, the tools for the mentioned above expansions are provided by the group representation theory, harmonic analysis on the group of rotations and the Peter-Weyl theorem. However, more caution and accuracy are needed when considering expansions for random functions, in particular, one needs to precise in what sense and under what conditions the corresponding series with random coefficients converge. These important issues are missing in physical literature, or at least they are not rigorously and clearly pronounced. It is also important to reveal in detail the nature and features of the random coefficients of the series expansions. Note that rigorous probabilistic framework is necessary to proceed with statistical inference methods.

Only recently the rigorous mathematical and probabilistic basis to underpin used in physical literature techniques appeared in several papers: [7], [8], [15]. In these papers the theory for random sections of fiber bundles over the sphere is developed with cosmological applications in mind.

In this paper we present an approach to derive spectral decompositions of tensor-valued random fields on the sphere from more group-theoretical point of view. We introduce specific components of a tensor field and under the conditions of weak isotropy and mean square

continuity define the representation of the rotation group by rotational shift operators acting on the Hilbert spaces generated by these components. Then spectral decompositions of the components are obtained as the sum of projections on the irreducible representation spaces, which expressed in terms of generalized spherical functions. We characterize the properties of random coefficients of the expansions obtained, including such an important question as conditions of Gaussianity.

Note that problems involving spectral analysis for random fields on the sphere arise in various applied areas such as geodesy, geophysics, planetary sciences, astronomy, cosmology, medical imaging etc.

The organization of the paper is as follows. We start in Section 2 with the brief overview of the representation theory of the rotation group, to that extend which will be needed in the further exposition. In Section 3 we review different approaches used to derive the spectral representation for scalar random fields on the sphere, with the particular attention paid to the approach based on representation of the rotation group with representation space, which is the Hilbert space generated by the random field. This last approach will be applied in Section 4 to derive spectral representations of vector and tensor fields.

2 Elements of the theory of the rotation group $SO(3)$ and its representations

We summarize briefly some important facts on the group $SO(3)$, which will be used in the following sections. In our exposition we prefer to follow to the classical books [6], [31], we will also indicate the connection to some other definitions and notions used in the literature, in particular, we will refer to yet another classical textbook [29]. However, many other excellent sources on the topic are available.

The group $SO(3)$ of special orthogonal transformations in R^3 consists of all rotations in R^3 about a fixed origin. This group has a convenient realization as the group of all 3×3 real matrices A such that $A^t A = I$ and $detA = 1$.

A number of parametric representations of rotations can be introduced. One important parametric form for the group of rotation uses the so-called Euler angles. Note that there exist different conventions how to define these angles. One of such conventions (e.g., [6]) is to define a rotation $g = g(\varphi_1, \theta, \varphi_2)$ as the product of three successive rotations: a rotation g_{φ_1} by an angle φ_1 around the z axis, then a rotation g_θ by an angle θ around the new x axis, then a rotation g_{φ_2} by an angle φ_2 around the new z axis. Correspondingly, the elements of the matrix of the rotation can be expressed explicitly in terms of the Euler angles $(\varphi_1, \theta, \varphi_2)$ (see, e.g., [6]).

Let V be a Hilbert space. A representation of a group G with a representation space V is a homomorphism $T : g \to T(g)$ of G into the space of bounded linear operators on V. Thus, the mapping T satisfies the conditions:

$$T(g_1)T(g_2) = T(g_1 g_2), \quad T(e) = E,$$

where e is an identity element of G, E is an identity operator. Note, that with each operator $T(g)$ one can associate its matrix $\{t_{ij}(g)\}$ (defined with respect to an orthonormal basis in V).

A representation T is said to be irreducible if there does not exist a proper subspace W of V, which is invariant under T. Otherwise, T is reducible. The representation T is said to be unitary if the operators $T(g)$, $g \in G$, are unitary with respect to the scalar product defined on V, i.e., for all $x, y \in V$, $g \in G$ we have

$$(T(g)x, T(g)y) = (x, y).$$

We formulate now the basic results on the resolution of a representation into irreducible representations (see, [6]).

Proposition 1 *Let $T : g \to T(g)$ be an unitary representation of the group of rotations in a (separable) Hilbert space V. Then there exist mutually orthogonal finite-dimensional subspaces $V_1, V_2, \ldots$, invariant with respect to T, in each of which the representation T is irreducible, and the space V is the orthogonal sum of these subspaces V_i. This means that every $x \in V$ can be expresses as a convergent series $x = \sum x_i$, $x_i \in V_i$, where convergence is meant with respect to the norm generated by the scalar product defined on V.*

Proposition 2 *Each irreducible representation of the group of rotations is defined by a number l, called the weight of representation, the corresponding invariant representation space D_l has dimension $2l+1$. The matrix of the representation (of weight l) corresponding to an arbitrary rotation $g = g(\varphi_1, \theta, \varphi_2)$ has, in the canonical basis, the form*

$$T^l(g) = \{T^l_{mn}(\varphi_1, \theta, \varphi_2)\}_{m,n=-l,\ldots,l},$$

where the elements are given by the following functions

$$T^l_{mn}(\varphi_1, \theta, \varphi_2) = e^{-im\varphi_1} P^l_{mn}(\cos\theta) e^{-in\varphi_2}. \tag{2.1}$$

Note that the matrix representation (2.1) is referred to the fixed canonical basis in the space D_l, which is formed by $2l+1$ orthogonal vectors (of dimension $2l+1$), each being the eigenvector with the eigenvalue $e^{-in\varphi}$, $n = -l, \ldots, l$, for the rotation around z axis by an angle φ.

The matrix elements $T^l_{mn}(\varphi_1, \theta, \varphi_2)$ are called the generalized spherical functions of the order l ([6]).

The functions $P^l_{mn}(z)$ appearing in the formula (2.1) can be represented in different ways. We give here their differential representation, which is also called the Rodrigues formula[1]:

$$\begin{aligned} P^l_{mn}(z) &= \frac{(-1)^{l-m} i^{m-n}}{2^l} \left[\frac{(l+n)!}{(l-m)!(l+m)!(l-n)!} \right]^{1/2} \\ &\times \; (1-z)^{-(n-m)/2} (1+z)^{-(n+m)/2} \frac{d^{l-n}}{dz^{l-n}} \left[(1-z)^{l-m} (1-z)^{l+m} \right]. \end{aligned} \tag{2.2}$$

Expressions for $P^l_{mn}(z)$ can be given in terms of hypergeometric functions, trigonometric functions, there exists also their integral representation. All this can be found, for example, in [31], [29], where the connection of $P^l_{mn}(z)$ with classical orthogonal polynomials and some other their properties are also reported.

In many cases one considers representations of a group G by shift operators in the linear space of functions defined on some space X, in particular, X can be equal to G itself. These shift operators acting on functions $f(g)$, $g \in G$, can be defined as

$$T(g_0) f(g) = f(g_0^{-1} g),$$

which corresponds to the case when G is considered as the group of left shifts, or one can define the action of the operator $T(g_0)$ as

$$T(g_0) f(g) = f(g g_0),$$

[1] The formula (2.2) is actually called the Rodrigues formula in [31] when $P^l_{mn}(z)$ is defined via (2.2) with i^{m-n} dropped, instead i^{m-n} is included in (2.1).

in the case when G is considered as the group of right shifts.

Let $L_2(G)$ be the Hilbert space of all functions on the compact group G, which are square integrable with respect to the Haar measure dg on G (this measure is invariant for compact groups). Considering the case $G = SO(3)$, we have the space of functions

$$f(g) = f(\varphi_1, \theta, \varphi_2),$$

for which the following integral exists

$$\int_G |f(g)|^2 dg = \frac{1}{8\pi^2} \int\int\int |f(\varphi_1, \theta, \varphi_2)|^2 \sin\theta d\varphi_1 d\theta d\varphi_2 < \infty; \tag{2.3}$$

the scalar product of $f_1(g)$ and $f_2(g)$ is defied as

$$(f_1, f_2) = \int f_1(g)\overline{f_2(g)}dg. \tag{2.4}$$

The transformation

$$T(g_0)f(g) = f(gg_0)$$

forms a unitary representation in the space $L_2(G)$ (called the (right) regular representation of the rotation group [6]). The irreducible representations into which it can be resolved are the representations in the subspaces of generalized spherical functions $T^l_{mn}(\varphi_1, \theta, \varphi_2)$ for a fixed l and m. Thus, the resolution of this representation into irreducible representations means that every function $f \in L_2(SO(3))$ can be expanded as a series in the functions $T^l_{mn}(\varphi_1, \theta, \varphi_2)$.

Proposition 3 *The set of the generalized spherical functions $T^l_{mn}(\varphi_1, \theta, \varphi_2)$ (l being an integer) forms a complete orthogonal system in the space of functions $L_2(SO(3))$.*

Note that the above statement is a particular case of the Peter-Weyl theorem, which is one of the most important results of the harmonic analysis on compact groups (see, e.g., [31]). We will return to this theorem later in this section and consider its stochastic version.

The multiplicative law of the group representation implies the following rule according to which the generalized spherical functions are added.

Proposition 4 *(Addition formula for the generalized spherical functions.)*

$$T^l_{mn}(g_1 g_2) = \sum_{s=-l}^{l} T^l_{ms}(g_1) T^l_{sn}(g_2). \tag{2.5}$$

Using the unitarity of the representation matrices, the above formula can also be written in the following form

$$T^l_{mn}(g_1 g_2^{-1}) = \sum_{s=-l}^{l} T^l_{ms}(g_1)\overline{T^l_{ns}(g_2)}. \tag{2.6}$$

Let us return to the definition of the Euler angles $(\varphi_1, \theta, \varphi_2)$. Another way to introduce them, most commonly used in physical literature, e.g., in quantum mechanics, is based on so-called zyz-convention about rotations (the introduced above is zxz-convention), when a rotation is defined as to be produced in the following three steps: a rotation g_{φ_1} by an angle φ_1 around the z axis, then a rotation g_θ by an angle θ around the new y axis, then a rotation g_{φ_2} by an angle φ_2 around the new z axis.

Correspondingly, within this approach, the matrix elements of the irreducible unitary representation of the weight l of the rotation group $SO(3)$ are represented by the Wigner D-functions $D^l_{mn}(\varphi_1, \theta, \varphi_2)$ (see, [30], [4], [29]). Proposition 3 holds for the functions D^l_{mn}, that is, every function from $L_2(SO(3))$ can be expanded in a series in terms of the Wigner D-functions.

Note that in quantum mechanics these functions appear to be the elements of the matrix representation of the rotation operator in the basis formed by the angular momentum eigenvectors. Wigner D-functions play an important role in various fields of modern physics including nuclear and molecular physics.

Both conventions about the definition of the Euler angles are closely related and the same rotation R can be achieved with a simple adjustment of angles: $R_{zyz}(\alpha, \beta, \gamma) = R_{zxz}(\alpha + \pi/2, \beta, \gamma - \pi/2)$ (see, [29]).

Correspondingly, there exists a simple relation between the functions T^l_{mn} and D^l_{mn} (see, [29]):

$$D^l_{mn}(\varphi_1, \theta, \varphi_2) = (-i)^{n-m} T^l_{mn}(\varphi_1, \theta, \varphi_2). \tag{2.7}$$

However, an advantage of the Wigner D-functions is that their factorized form is given in terms of the real Wigner d-functions $d^l_{mn}(\theta)$ and the complex exponentials:

$$D^l_{mn}(\varphi_1, \theta, \varphi_2) = e^{-im\varphi_1} d^l_{mn}(\cos\theta) e^{-in\varphi_2}, \tag{2.8}$$

where d^l_{mn} are equal to P^l_{mn} with the factor $(-i)^{n-m}$ dropped (cf. (2.1)-(2.2)).

As we have seen above the group representation theory (outlined here for a particular case of the group $G = SO(3)$) provides a tool – the Peter-Weyl theorem – for decomposition of $L_2(G)$ space (endowed with the Haar measure) into an orthogonal sum of finite dimensional spaces and, correspondingly, the decomposition of functions from $L_2(G)$ into the sum of their projections on these spaces which can be represented in terms of appropriate basis functions.

Our main interest is in extension of this result which allows for decomposition of random functions. One can observe immediately that such a decomposition can be derived for random functions $X(g)$, $g \in G$, whose trajectories $g \to X(g)$ are P-a.s. square integrable w.r.t. the Haar measure (that is, belong to $L_2(G)$), corresponding decomposition must be understood in the $L_2(G)$-sense, P-a.s. However, more profound results can be obtained with the assumption of isotropy, that is, invariance in law of a random function $X(y)$, $y \in Y$, under the action of a group G (with G being a topological compact group acting on Y, Y can be equal to G itself). In such a case one can formulate a stochastic version of the Peter-Weyl theorem (see, [26]) and the decomposition of $L_2(G)$ translates easily into construction of spectral representations for isotropic random functions. To be more precise, for the case of the rotation group $SO(3)$ the following result can be stated as a consequence of the stochastic Peter-Weyl theorem (see, [20], [26]).

Proposition 5 *Let $X(g)$, $g \in G = SO(3)$, be a square integrable ($L^2(P(d\omega))$), strictly isotropic random field, that is, finite-dimensional distributions of*

$$\{X(g_1), \ldots, X(g_k)\} \quad \text{and} \quad \{X(gg_1), \ldots, X(gg_k)\}$$

are the same $\forall k$, $\forall g$, $g_1, \ldots, g_k \in G$. Then

$$X(g) = X(\varphi, \theta, \psi) = \sum_l \sum_{m,n} a_{lmn} \sqrt{\frac{2l+1}{8\pi^2}} D^l_{mn}(\varphi, \theta, \psi), \tag{2.9}$$

both in $L^2(P(d\omega)\times dg)$ and pointwise in $L^2(P(d\omega))$, where dg is the Haar measure on $SO(3)$, and

$$a_{lmn} = \int_G X(g)\sqrt{\frac{2l+1}{8\pi^2}}\overline{D^l_{mn}}(g)dg. \tag{2.10}$$

Remark 1 The functions $D^l_{mn}(g)$ (as well as $T^l_{mn}(g)$) are orthogonal but not orthonormal. The orthogonality relation for $D^l_{mn}(g)$ reads

$$\int_G D^l_{mn}(g)\overline{D^{l'}_{m'n'}}(g)dg = \frac{8\pi^2}{2l+1}\delta_{ll'}\delta_{mm'}\delta_{nn'}.$$

This explains the appearance of the factor $\sqrt{\frac{2l+1}{8\pi^2}}$ in (2.9)-(2.10).

Remark 2 In what follows we will refer to the above result, however we will also consider the representation of $SO(3)$ via (rotational) shift operators defined on the Hilbert space generated by a random field, which gives rise to alternative derivation of the spectral representation (2.9) and allows to treat the coefficients from the different point of view.

3 Spectral representation of scalar fields

In this section we review different approaches used to derive the spectral representation for scalar random fields on the sphere.

In what follows we will suppose that we are given the probability space $(\Omega, \mathcal{F}, P)$.

Let $L^2(P(d\omega))$ be the Hilbert space of random variables Y such that $E|Y|^2 < \infty$, with inner product $(Y_1, Y_2) = E(Y_1\overline{Y_2})$; a point in the unit sphere S_2 be denoted $t \equiv (1, \theta, \varphi)$; $X(t) = X(t,\omega)$ be mean square continuous zero mean random field on S_2, $L^2_X(P(d\omega))$ be the Hilbert space generated by $X(t)$, G be the group of rotations, $G = SO(3)$.

1. First approach (which we present here following to [11] and [24]) is based on the theory of representations of the rotation group. The representation of the rotation group is defined by (rotational) shift transformations on the Hilbert space generated by the random field $X(t)$.

$X(t)$, $t \in S_2$, can be regarded as a random field on G:

$$X(t) \equiv X(g_t e_0), \quad t \equiv g_t e_0, \quad g_t \in G, \quad g_t = g(\varphi + \frac{\pi}{2}, \theta, \varphi_2), \tag{3.1}$$

where e_0 denotes a unit vector along the polar axis. (We suppose we have fixed the Cartesian coordinate system with the unit vectors (e_x, e_y, e_z), with $e_z = e_0$ being the unit vector along the polar axis of the sphere S_2 and the origin is at the center of the sphere.)

The scalar field (3.1) is independent of the third Euler angle or of the rotation around t.

The field $X(t)$ is said to be isotropic in the wide sense, if the correlation function $R(t_1, t_2) = E(X(t_1)\overline{X(t_2)})$ is invariant under arbitrary rotation:

$$R(t_1, t_2) = R(gt_1, gt_2), g \in G. \tag{3.2}$$

This implies that $R(t_1, t_2)$ is a function only of $\cos\theta$, where θ is the angle between the unit vectors t_1 and t_2 with spherical coordinates (θ_1, φ_1) and (θ_2, φ_2),
$\cos\theta = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\varphi_1 - \varphi_2)$.

Define the rotational transformation of the field $X(t)$:

$$S^g : X(t) \to X(g^{-1}t),\ g \in G. \tag{3.3}$$

This transformation induces a transformation on random variables from L^2_X:

$$U^g : Y \to U^g Y,\ g \in G,\ Y \in L^2_X. \tag{3.4}$$

From invariance (3.2) it follows that the scalar product in L^2_X, that is the covariance of $Y, Z \in L^2_X$ is invariant under U^g:

$$E(U^g Y \overline{U^g Z}) = E(Y\overline{Z}), \tag{3.5}$$

which implies that U^g is a unitary operator on L^2_X. Also, we have the group properties for U^g:

$$U^{g_1}U^{g_2} = U^{g_1 g_2},\quad (U^g)^{-1} = U^{g^{-1}},\quad U^e = 1. \tag{3.6}$$

This means that U^g, $g \in G$, gives the unitary representation of the rotation group G in the Hilbert space L^2_X. From m.s. continuity of $X(t)$ it follows that U^g is continuous w.r.t. g.

In particular, $U^g X(t) = X(g^{-1}t)$, therefore, we can write:

$$X(t) = X(ge_0) = U^{g^{-1}}X(e_0). \tag{3.7}$$

Denote by H the subgroup of rotations around the polar axis e_0 and by H_t the subgroup of rotations around the vector t. Then for the scalar filed $X(t)$ we have

$$U^h X(t) = X(t),\ h \in H_t,\ U^h X(e_0) = X(e_0),\ h \in H. \tag{3.8}$$

By the representation theory of the rotation group, the representation space L^2_X for U^g can be decomposed into the sum of the irreducible spaces. Correspondingly, the vector $X(e_0)$ from L^2_X can be decomposed into the sum of the vectors of this orthogonal irreducible spaces. Denote by $D_l(\Omega)$ an irreducible space of the weight l representation for U^g. Denote the canonical basis for $D_l(\Omega)$ by

$$Z_l^m,\ m = -l, \ldots, l; \tag{3.9}$$

the orthogonality relations are:

$$E(Z_l^m \overline{Z_{l'}^{m'}}) = \delta_{ll'}\delta_{mm'}. \tag{3.10}$$

By (3.8), $X(e_0)$ has only 0-th canonical component for each l, that is, its irreducible decomposition can be written only in terms of Z_l^0:

$$X(e_0) = \sum_{l=0}^{\infty} \tilde{F}_l Z_l^0, \tag{3.11}$$

where the series converges in $L^2(P(d\omega))$, the expansion coefficients can be given:

$$\tilde{F}_l = E(X(e_0)\overline{Z_l^0}),\ l = 0, 1, 2, \ldots \tag{3.12}$$

To obtain $X(t)$ we apply $U^{g^{-1}}$:

$$X(t) = \sum_{l=0}^{\infty} \tilde{F}_l U^{g^{-1}} Z_l^0. \tag{3.13}$$

The final step is to represent $U^{g^{-1}}Z_l^0$ in terms of the canonical basis Z_l^m. We know the form of the matrix of the representation in the canonical basis, therefore, we can write:

$$U^{g^{-1}}Z_l^0 = \sum_{m=-l}^{l} T_{0m}^l(g^{-1})Z_l^m = \sqrt{\frac{4\pi}{2l+1}} \sum_{m=-l}^{l} Y_l^m(\theta, \varphi) Z_l^m, \tag{3.14}$$

substituting this into (3.13) we obtain the spectral representation of the field $X(t)$:

$$X(t) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} F_l Y_l^m(\theta, \varphi) Z_l^m \tag{3.15}$$

in $L^2(P(d\omega))$, here we have denoted $F_l = \tilde{F}_l \sqrt{\frac{4\pi}{2l+1}}$.

For the covariance function we obtain:

$$\begin{aligned} R(\theta) &= E(X(t_1)\overline{X(t_2)}) = \sum_{l=0}^{\infty} |F_l|^2 \sum_{m=-l}^{l} Y_l^m(\theta_1, \varphi_1)\overline{Y_l^m(\theta_2, \varphi_2)} \\ &= \frac{1}{4\pi} \sum_{l=0}^{\infty} (2l+1)|F_l|^2 P_l(\cos\theta), \end{aligned} \tag{3.16}$$

applying the addition formula for spherical harmonics $Y_l^m(\theta, \varphi)$

$$P_l(\cos\theta) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_l^m(\theta_1, \varphi_1)\overline{Y_l^m(\theta_2, \varphi_2)}, \tag{3.17}$$

where P_l is the Legendre polynomial.

Remark 3 It is interesting to note that within the described above constructive approach for derivation of the spectral representation we obtain the spherical harmonics coefficients in the factorized form $F_l Z_l^m$ (see, (3.15)) with random variables Z_l^m being uncorrelated (orthonormal) in a way (3.10) and the factor F_l is common for all Z_l^m, $m = -l, ..., l$. Note that $|F_l|^2$ is called the angular power spectrum and $F_l Z_l^m$ is called the random spectrum in [24] (or multipole coefficients, in physical literature).

2. The spectral representation of a random field on a sphere can be obtained as a particular case of the stochastic Peter-Weyl theorem (see, [20], [26] and the end of Section 2). Indeed, Proposition 5 gives the spectral decomposition for a square integrable isotropic random field $X(g)$ on the group $SO(3)$. If we consider the restriction of $X(g)$ on the quotient space $S_2 = SO(3)/SO(2)$, then in the representation (2.9) the inner double sum will reduce to the single sum over $m = -l, ..., l$, the functions $D^l_{mn}(\varphi, \theta, \psi)$ will reduce to $D^l_{m0}(\varphi, \theta, \psi) = \sqrt{\frac{4\pi}{2l+1}} Y_l^m(\theta, \varphi)$, and the coefficients a_{lmn} given by the formula (2.10) will simplify to

$$a_{lmn} = \begin{cases} 0, & n = 0 \\ \sqrt{2\pi} a_{lm}, & n \neq 0 \end{cases}$$

with a_{lm} given by

$$a_{lm} = \int_{S_2} X(\theta, \varphi)\overline{Y_l^m(\theta, \varphi)} \sin\theta \; d\varphi \; d\theta, \tag{3.18}$$

that is, the resulting representation appears in the form

$$X(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_l^m(\theta, \varphi), \tag{3.19}$$

where the random coefficients are given by (3.18).

Remark 4 Comparing (3.19) and (3.15) we see $a_{lm} = F_l Z_l^m$, that is we can treat the random coefficients in the spectral decomposition from different points of view. Note also that $E|a_{lm}|^2 = F_l^2$ is angular power spectrum, which is more commonly denoted as C_l.

3. We would like to recall that the very elegant derivation of the spectral representation of a random field on the sphere can be given via application of classical results from analysis and probability theory, just in three steps: (i) from the Funk-Hecke theorem one can enjoy a complete set of eigenvalues and orthonormal eigenfunctions for the covariance function of a mean square continuous homogeneous isotropic random field on the sphere, hence (ii) Mercer theorem allows to write down the decomposition of the covariance function and finally (iii) Karhunen theorem implies the spectral decomposition of the field itself (for more detail see, e.g., [13], [32]).

4 Spectral representations of vector and tensor fields

Let now $X(t)$, $t \in S_2$, be a vector random field, that is, given a fixed coordinate system, the filed $X(t)$ at each point $t \in S_2$ can be represented by its coordinates $(X_1(t), X_2(t), X_3(t))$, which will change according to the certain rules, when the coordinate system is changed. We will also suppose that the field has zero mean.

We want to consider the field $X(t)$, which is a vector function of two variables $(\theta, \varphi) \in S_2$, as being defined on the group of rotations G, moreover we want to represent the field by a system comprising three functions of three Euler angles $(\varphi_1, \theta, \varphi_2)$, functions of rotations, which transform into themselves under rotations. Thus, rotational shift transformations of each of such functions into itself will give rise to a representation of the group $SO(3)$ and analogously to Section 3 will lead to corresponding spectral decompositions. To obtain such functions we will use ideas from [6].

Fix the Cartesian coordinate system with the unit vectors (e_x, e_y, e_z), with $e_z = e_0$ being the unit vector along the polar axis of the sphere S_2 and the origin at the center of the sphere. Let $(e_r, e_\theta, e_\varphi)$ be the basis unit vectors for the polar (spherical) coordinate system, this triad is associated with the point $(1, \theta, \varphi)$ of the unit sphere and constitutes the so-called 'moving frame'. With each rotation we can associate, however, the triad related to an element g of the rotation group. We associate with the point P_0, the 'North Pole' of the sphere, the triad (e_x, e_y, e_z) and this will correspond to the identity rotation e. Each rotation g transforms the triad at P_0 into a triad (e_1, e_2, e_3) and places it at some point P on the sphere, that is, the element g of the group $SO(3)$ determines the triad and the point which it is associated with. The triad corresponding to the rotation $g = g(\varphi_1, \theta, 0)$ is $(-e_\varphi, e_\theta, e_r)$ and it is placed at point $P(\varphi, \theta)$, where $\varphi = \frac{\pi}{2} - \varphi_1$, and to the rotation $g = g(\varphi_1, \theta, \varphi_2)$ there correspons the triad (e_1, e_2, e_3) given by

$$\begin{aligned} e_1 &= -e_\varphi \cos\varphi_2 - e_\theta \sin\varphi_2, && (4.1)\\ e_2 &= -e_\varphi \sin\varphi_2 + e_\theta \cos\varphi_2, && (4.2)\\ e_3 &= e_r. && (4.3) \end{aligned}$$

Now considering the vector $X(P)$ at point P we resolve it with respect to (e_1, e_2, e_3) to obtain the components $(X_1(g), X_2(g), X_3(g))$. The component

$$X_3(g) = X_r(\varphi_1, \theta, \varphi_2) = X_r(\theta, \varphi), \quad \varphi = \varphi_2 - \pi/2, \tag{4.4}$$

is normal to the surface of the sphere at point $P(\theta, \varphi)$ and does not depend on the third Euler angle φ_2, from the other two components (lying at the tangent plane at point P) we form two

complex components

$$\begin{aligned} X_+(g) &= X_+(\varphi_1, \theta, \varphi_2) = X_1(g) + iX_2(g), \\ X_-(g) &= X_-(\varphi_1, \theta, \varphi_2) = X_1(g) + iX_2(g). \end{aligned}$$

These complex components can be represented as

$$X_+(g) = e^{i\varphi_2}[-X_\varphi(\theta, \varphi) + iX_\theta(\theta, \varphi)], \tag{4.5}$$

$$X_-(g) = e^{-i\varphi_2}[-X_\varphi(\theta, \varphi) - iX_\theta(\theta, \varphi)], \tag{4.6}$$

where $X_\varphi(\theta, \varphi)$, $X_\theta(\theta, \varphi)$ are the components of the vector field in the polar coordinate system, $\varphi = \frac{\pi}{2} - \varphi_1$.

With the above three functions we have defined the vector field on the rotation group:

$$X(\varphi_1, \theta, \varphi_2) = X(g) = (X_+(g), X_-(g), X_r(g)).$$

With each rotation g_0 the normal component $X_r(g) = X_r(P)$ at point $P(\varphi, \theta)$ transforms into the normal component at the point $g_0^{-1}g = g_0^{-1}P$ and does not depend on the third Euler angle, that is, on the rotation in the tangent plane.

We can consider the normal component (4.4) and the component of the field in the tangent plane, which is represented by two functions (4.5)-(4.6), separately.

Remark 5 Note that in applications vector fields on the sphere are usually considered as vectors in the tangent plane at each point of the sphere. Typical example of vector fields on the sphere that arise in practice are electromagnetic fields or wind velocity.

The notion of isotropy (in weak or second-order sense) we will define componentwise, as isotropy property for the components (4.4)-(4.6), that is, the covariance functions

$$R_+(g_1, g_2) = EX_+(g_1)\overline{X_+(g_2)}, \tag{4.7}$$

$$R_-(g_1, g_2) = EX_-(g_1)\overline{X_-(g_2)}, \tag{4.8}$$

$$R_r(g_1, g_2) = EX_r(g_1)\overline{X_r(g_2)} \tag{4.9}$$

are invariant with respect to rotations $g \in SO(3)$:

$$R_i(g_1, g_2) = R_i(gg_1, gg_2), \quad \forall g, \quad i = \pm, r, \tag{4.10}$$

which implies $R_i(g_1, g_2) = R_i(g_1^{-1}g_2)$, $i = \pm, r$.

Remark 6 We believe that defining the notion of isotropy in such a specific way, that is, isotropy for the components taken in the moving basis (4.1)-(4.3) is quite reasonable for vector fields on the sphere. This will be quite analogous to the notion of isotropic vector fields introduced in early works on turbulence (see, e.g., [27], [22] and references therein). Recall that in those works isotropy for a vector field in strict or wide sense is defined as a property which prescribes that probabilistic characteristics such as probability distributions for the values of a field in a system of points (strict sense) or just correlations (wide sense) are invariant under translations (that is, motions, which include rotations, shifts and reflections) of the the system of points performed simultaneously with the same movement of the coordinate system. In particular, for the correlation tensor of the vector field this definition entails the following property:

$$B(t) = G^* B(Gt) G,$$

where G is a matrix of the corresponding transformation and G^* is its transpose. (In some studies a field with the above property of its correlation tensor is also called isotropic in a vector sense.) Considering random fields on the sphere we are restricted to the rotations only and define the isotropy as invariance of probabilistic characteristics under rotations. In our approach we consider the vector field on the sphere defined at each point of the sphere via its components in the moving basis, which is associated with a rotation g. Therefore, these movements (rotations) of the coordinate system are already incorporated into such a representation of a vector field. And each component of the field transforms into itself under rotation. That is why it looks quite natural to define isotropy via the relations (4.10). Moreover, this will give us the possibility to define the scalar product, which will be invariant under rotations, in the corresponding Hilbert spaces of random variables, and, therefore, we can parallel all the constructions and reasonings of the previous section, where scalar fields have been considered.

First, as we noticed above, the normal component of the field $X_r(\theta, \varphi)$ does not depend on the third Euler angle and can be treated completely in the same manner as the scalar field, and the spectral representation of the form (3.15) can be written, that is, the representation via the spherical harmonics $Y_l^m(\theta, \varphi)$, and the corresponding representation of the correlation function in the form (3.16).

Consider now the tangent component of the filed given by two functions (4.5)-(4.6). To obtain the spectral representation we will follow the lines of Section 3.

Let $L^2_{X_+}$ be the Hilbert space generated by $X_+(g)$.

Define the rotational (or shift) transformation:

$$S^g : X_+(g_0) \to X_+(g^{-1}g_0), \; g, g_0 \in G. \tag{4.11}$$

This transformation induces a transformation on random variables from $L^2_{X_+}$:

$$U^g : Y \to U^g Y, \; g \in G, \; Y \in L^2_{X_+}. \tag{4.12}$$

From invariance (4.7) it follows that the scalar product in $L^2_{X_+}$, that is the covariance of $Y, Z \in L^2_{X_+}$ is invariant under U^g:

$$E(U^g Y \overline{U^g Z}) = E(Y\overline{Z}), \tag{4.13}$$

which implies that U^g is a unitary operator on $L^2_{X_+}$. Also, we have the group properties for U^g: $U^{g_1}U^{g_2} = U^{g_1 g_2}$, $(U^g)^{-1} = U^{g^{-1}}$, $U^e = 1$. This means that U^g, $g \in G$, gives the unitary representation of the rotation group G in the space $L^2_{X_+}$. Assuming mean square continuity of $X_+(t)$, we obtain that U^g is continuous w.r.t. g.

In particular, $U^g X_+(g_0) = X_+(g^{-1}g_0)$, therefore, we can write:

$$X_+(g) = X_+(ge) = U^{g^{-1}} X_+(e), \tag{4.14}$$

where e is identity element of G, and

$$X_+(e) = X_+(e_0) = X_x(e_0) + iX_y(e_0) \tag{4.15}$$

that is, it reduces to the component of the field at the point $e_0 = P_0$ (the pole of our sphere).

Denote by $H_{e_0}(\varphi)$ the subgroup of rotations around the polar axis e_0 and by $H_g(\varphi)$ the subgroup of rotations around the vector ge_0. Then for the filed $X_+(g)$ we have

$$U^h X_+(g) = e^{i\varphi} X_+(g), \; h \in H_g(\varphi), \; U^h X_+(e_0) = e^{i\varphi} X_+(e_0), \; h \in H_{e_0}(\varphi). \tag{4.16}$$

The representation space $L^2_{X_+}$ for U^g can be decomposed into the sum of the irreducible spaces. Correspondingly, the vector $X_+(e)$ from $L^2_{X_+}$ can be decomposed into the sum of the vectors of this orthogonal irreducible spaces. Denote by $D_l(\Omega)$ an irreducible space of the weight l, let

$$Z^+_{lm},\ m = -l, \dots, l, \tag{4.17}$$

be the canonical basis for $D_l(\Omega)$, that is the basis composed from the eigenvectors of U^h, $h \in H_{e_0}(\varphi)$ (rotation about e_0) corresponding to the eigenvalues $e^{i\varphi m}$, for these vectors the orthogonality relations hold:

$$E(Z^+_{lm}\overline{Z^+_{l'm'}}) = \delta_{ll'}\delta_{mm'}. \tag{4.18}$$

As can be seen from (4.16), $X(e_0)$ has only '1'-st component non-zero in the basis (4.17), for each l, therefore, its irreducible decomposition can be written only in terms of Z^+_{l1}:

$$X_+(e_0) = \sum_{l=0}^{\infty} F^+_l Z^+_{l1}, \tag{4.19}$$

where the series converges in $L^2(P(d\omega))$, the expansion coefficients can be given:

$$F^+_l = E(X_+(e_0)\overline{Z^+_{l1}}),\ l = 0, 1, 2, \dots \tag{4.20}$$

To obtain $X(g)$ we apply $U^{g^{-1}}$:

$$X_+(g) = \sum_{l=0}^{\infty} F^+_l U^{g^{-1}} Z^+_{l1}. \tag{4.21}$$

Now it is left to notice that

$$U^{g^{-1}} Z^+_{l1} = \sum_{m=-l}^{l} T^l_{1m}(g^{-1}) Z^+_{lm}, \tag{4.22}$$

substituting this into (4.21) we obtain the spectral representation of the field $X_+(g)$:

$$X_+(g) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} F^+_l T^l_{1m}(g^{-1}) Z^+_{lm} \tag{4.23}$$

in $L^2(P(d\omega))$.

For the covariance function we obtain:

$$\begin{aligned} R_+(g_1, g_2) &= E(X_+(g_1)\overline{X_+(g_2)}) = \sum_{l=0}^{\infty} |F^+_l|^2 \sum_{m=-l}^{l} T^l_{1m}(g_1^{-1})\overline{T^l_{1m}(g_2^{-1})} \\ &= \sum_{l=0}^{\infty} |F^+_l|^2 T^l_{11}(g_1^{-1} g_2), \end{aligned} \tag{4.24}$$

applying the addition formula (2.6) and the orthogonality relations (4.18).

Analogously we obtain the spectral representation of the field $X_-(g)$:

$$X_-(g) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} F^-_l T^l_{-1m}(g^{-1}) Z^-_{lm} \tag{4.25}$$

in $L^2(P(d\omega))$, where r.v. Z^-_{lm} satisfy the orthogonality relations $E(Z^-_{lm}\overline{Z^-_{l'm'}}) = \delta_{ll'}\delta_{mm'}$.

For the covariance function we obtain:

$$\begin{aligned} R_-(g_1,g_2) &= E(X_-(g_1)\overline{X_-(g_2)}) = \sum_{l=0}^{\infty} |F_l^-|^2 \sum_{m=-l}^{l} T^l_{-1,m}(g_1^{-1})\overline{T^l_{-1,m}(g_2^{-1})} \\ &= \sum_{l=0}^{\infty} |F_l^-|^2 T^l_{-1,-1}(g_1^{-1}g_2) \end{aligned} \tag{4.26}$$

Let us summarize the above reasonings in the following theorem.

Theorem 1 *Let $X(t)$, $t=(\theta,\varphi)\in S_2$, be a zero-mean vector random field on the unit sphere and $(X_\varphi(\theta,\varphi), X_\theta(\theta,\varphi), X_r(\theta,\varphi))$ be its components in spherical coordinate system. Introducing a specific local coordinate system (4.1)-(4.3), which at every point of the sphere is related to an element of the group of rotations, consider the representation of the field X by the components defined on the group of rotation: two complex components $X_+(g)$ and $X_-(g)$ given by (4.5)-(4.6) and the normal $X_r(g)=X_r(t)$. Suppose that these functions $X_+(g)$, $X_-(g)$ and $X_r(g)$ are weakly (second-order) isotropic and mean square continuous. Then the functions $X_+(g)$ and $X_-(g)$ can be expanded in the series of generalized spherical harmonics (4.23) and (4.25), respectively, with uncorrelated coefficients, convergence is meant pointwise in $L^2(P(d\omega))$. Corresponding covariance functions have series representations (4.24) and (4.26). The normal component can be treated and expanded as a scalar random field.*

Let us turn now to the tensor fields on the unit sphere. We will consider the case of tensors of second rank.

To find the expansions for tensor fields we can apply the reasonings similar to the above ones and represent the tensor fields by means of components (functions on the group $SO(3)$) which transform into itself under rotations and which will be multiplied by $e^{\pm im\varphi}$, $m=0,1,2$, under rotations about the axis normal to the surface of the sphere.

To obtain such components we note that components of a tensor of the second rank are transformed under a rotation in the same way as a product of the components of two vectors. Thus, for tensor random field on the sphere $X(t)$, $t\in S_2$, we come to the following nine components – functions of rotations $g=g(\varphi_1,\theta,\varphi_2)$:

$$X_{rr}, \quad X_{\varphi\varphi}+X_{\theta\theta}\pm i\,(X_{\varphi\theta}-X_{\theta\varphi})\,, \tag{4.27}$$

$$(-X_{\varphi r}-iX_{\theta r})\,e^{i\varphi_2}, (-X_{r\varphi}-iX_{r\theta})\,e^{i\varphi_2}, \tag{4.28}$$

$$(-X_{\varphi r}+iX_{\theta r})\,e^{-i\varphi_2}, (-X_{r\varphi}+iX_{r\theta})\,e^{-i\varphi_2}, \tag{4.29}$$

$$X_{\varphi\varphi}-X_{\theta\theta}+i\,(X_{\theta\varphi}-X_{\varphi\theta})\,e^{2i\varphi_2}, \tag{4.30}$$

$$X_{\varphi\varphi}-X_{\theta\theta}-i\,(X_{\theta\varphi}-X_{\varphi\theta})\,e^{-2i\varphi_2}. \tag{4.31}$$

With cosmological applications in mind, we restrict our consideration to the tangential counterpart of the field, that is, we will be interested in the components (4.30)-(4.31) only. Moreover, we will suppose that this tangential counterpart (the components of the tensor relative to the basis in the tangent plane) forms a symmetric trace-free tensor, that is, $X_{\varphi\varphi}+X_{\theta\theta}=0$, $X_{\theta\varphi}=X_{\varphi\theta}$. In such a case the components (4.30)-(4.31) reduce to the form

$$2\,(X_{\varphi\varphi}+iX_{\theta\varphi})\,e^{2i\varphi_2} \qquad \equiv A(g), \tag{4.32}$$

$$2\,(X_{\varphi\varphi}-iX_{\theta\varphi})\,e^{-2i\varphi_2} \qquad \equiv B(g). \tag{4.33}$$

Now to derive the spectral representation for the random functions (4.32)-(4.33) we apply the same scheme as used above for the components (4.5)-(4.6) of a vector field.

We suppose that the random functions $A(g)$ and $B(g)$ are weakly (or second-order) isotropic, that is, their covariance functions satisfy

$$R_i(g_1, g_2) = R_i(gg_1, gg_2), \quad \forall g, g_1, g_2 \in SO(3), \quad i = A, B, \tag{4.34}$$

which implies $R_i(g_1, g_2) = R_i(g_1^{-1} g_2)$, $i = A, B$.

Then we consider the representations of the rotation group by rotational transformations defined on the Hilbert spaces L^2_A and L^2_B generated by the random functions $A(g)$ and $B(g)$, which will be: (1) unitary due to the invariance of the scalar product under rotational transformations (see, (4.34)), (2) continuous under the assumption of mean square continuity of our random field.

We notice also that under rotations by an angle φ about the axis normal to the surface of the sphere (that is, around e_r) the functions $A(g)$ and $B(g)$ will multiply by $e^{i2\varphi}$ and $e^{-i2\varphi}$ respectively.

The representation spaces L^2_A and L^2_B can be decomposed into the orthogonal sums of the irreducible spaces of weight l, D_l^A and D_l^B, introducing the canonical bases $\{Z^A_{lm}\}$ and $\{Z^B_{lm}\}$, $m = -l, ..., l$, in D_l^A and D_l^B correspondingly, we come to the representations for the functions $A(g)$ and $B(g)$.

Namely, we obtain:

$$A(g) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_l T^l_{2m}(g^{-1}) Z^A_{lm}, \tag{4.35}$$

$$B(g) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} B_l T^l_{-2,m}(g^{-1}) Z^B_{lm}, \tag{4.36}$$

where the series converge in $L^2(P(d\omega))$, r.v.'s Z^A_{lm} and Z^B_{lm} satisfy the orthogonality relations

$$E(Z^i_{lm} \overline{Z^i_{l'm'}}) = \delta_{ll'} \delta_{mm'}, \quad i = A, B, \tag{4.37}$$

the coefficients A_l and B_l are given by the formulas

$$A_l = E(A(e_0) \overline{Z^A_{2l}}), \quad B_l = E(B(e_0) \overline{Z^B_{-2,l}}), \ l = 0, 1, 2, \ldots \tag{4.38}$$

For the correlation functions we obtain

$$R_A(g_1^{-1} g_2) = \sum_{l=0}^{\infty} |A_l|^2 T^l_{22}(g_1^{-1} g_2), \tag{4.39}$$

$$R_B(g_1^{-1} g_2) = \sum_{l=0}^{\infty} |B_l|^2 T^l_{-2,-2}(g_1^{-1} g_2). \tag{4.40}$$

We come to the following theorem.

Theorem 2 *Let $X(t)$, $t = (\theta, \varphi) \in S_2$, be a zero-mean tensor-valued random field on the unit sphere. Suppose that the rank of the tensor is 2, its components are given with respect to polar coordinate system and tangential counterpart of this tensor forms a symmetric trace-free tensor. Consider the representation of the tangential counterpart of the field X by two complex components defined on the group of rotation: $A(g)$ and $B(g)$ given by (4.32)-(4.33) and suppose that these functions are weakly (second-order) isotropic and mean square continuous. Then the functions $A(g)$ and $B(g)$ can be expanded in the series of generalized spherical functions (4.35) and (4.36), respectively, with uncorrelated coefficients and convergence is meant pointwise in $L^2(P(d\omega))$. Corresponding covariance functions have series representations (4.39) and (4.40).*

Remark 7 We restrict ourselves here to consideration of representations for the random functions (4.32)-(4.33). Note that the same representation holds for more general functions (4.30)-(4.31). The functions (4.27) can be expanded in terms of the usual spherical harmonics $Y_m^l(\theta, \varphi)$, the functions (4.28) – in terms of T_{1m}^l, and the functions (4.29) – in terms of $T_{-1,m}^l$. Moreover, in the similar way the representation of tensor fields of higher ranks then 2 can be obtained (see also Remark 9 below).

Let us consider more carefully the random coefficients Z_{lm}^+, Z_{lm}^- and Z_{lm}^A, Z_{lm}^B of the expansions (4.23)-(4.25) and (4.35)-(4.36). First immediate properties of these random variables we summarize in the following lemma.

Lemma 1 *Under the condition of mean square continuity and weak isotropy of the random function $X_+(g)$, the following statements hold:*

(1) the random variables Z_{lm}^+ are uncorrelated with zero mean and unit variance:

$$E(Z_{lm}^+ \overline{Z_{l'm'}^+}) = \delta_{ll'}\delta_{mm'};$$

(2) for all l, $\overline{Z_{lm}^+} = (-1)^{l-m} Z_{l,-m}^+$, and, therefore, $E(Z_{lm}^+ Z_{l'm'}^+) = 0$ for all $(l,m) \neq (l',m')$;

(3) for all l, $g \in G$, $Z_{lm}^+ \stackrel{d}{=} \sum_{s=-l}^{l} T_{ms}^l(g) Z_{lm}^+$, in particular, $Z_{lm}^+ \stackrel{d}{=} e^{im\varphi} Z_{lm}^+ \ \forall \varphi$.

The same properties hold for the random variables Z_{lm}^- and Z_{lm}^A, Z_{lm}^B under the conditions of mean square continuity and isotropy of the corresponding fields.

Proof. All three facts follow from the very definition of the random variables Z_{lm}^+, which have been chosen as canonical bases in the spaces D_l. For every l, $\left\{Z_{lm}^+, \ \ m = -l, ..., l\right\}$ are orthonormal bases in orthogonal spaces D_l, which means that (1) holds. Moreover, $\left\{Z_{lm}^+, \ \ m = -l, ..., l\right\}$ constitute the canonical bases, for all l, therefore $U^h Z_{lm}^+ = e^{im\varphi} Z_{lm}^+$, h being the rotation around e_0 by an angle φ. Taking into account that U^g is an unitary operator for which the contravariant basis $\left\{Z_l^{+(m)}, \ \ m = -l, ..., l\right\}$ coincides with $\left\{\overline{Z_{lm}^+}, \ \ m = -l, ..., l\right\}$ and using the relation between the vectors of canonical basis and its cotravariant basis: $Z_l^{+(m)} = (-1)^{l-m} Z_{l,-m}^+$ (see [14]), we come to the statement (2) of the lemma. Further, we know that for every l, $T^l(g) = \left\{T_{ms}^l(g), \ \ m, n = -l, ..., l\right\}$ are the matrices of the representation U^g in the spaces D_l, with respect to the canonical basis in D_l, that is with respect to $\left\{Z_{lm}^+, \ \ m = -l, ..., l\right\}$. This implies the equality $U^g Z_{lm}^+ = \sum_{s=-l}^{l} T_{ms}^l(g) Z_{lm}^+$. On the other hand, isotropy of the field $X_+(g)$ entails the equality $U^g Z_{lm}^+ \stackrel{d}{=} Z_{lm}^+ \ \forall l, \ \forall m$. This proves the statement (3) of the lemma. Since all the random variables Z_{lm}^+, Z_{lm}^-,Z_{lm}^A, Z_{lm}^B have the same nature, the properties stated in the lemma are common for all of them.

Remark 8 Considering $2l + l$-dimensional vectors composed from the random variables Z_{lm}^+ with fixed l, that is the vectors $Z_l^+ = \left\{Z_{lm}^+, \ \ m = -l, ..., l\right\}$, we can reformulate the statement (3) of Lemma 1 in the following form: $Z_l^+ \stackrel{d}{=} T^l(g) Z_l^+$. Statement (1) of Lemma 1 can be reformulated in a slightly more general form as $E(Z_l^+ \overline{Z_l^+}) = I_{2l+1}$ and $E(Z_l^+ \overline{Z_{l'}^+}) = 0$, $l \neq l'$, where I_{2l+1} is $(2l+1) \times (2l+1)$ identity matrix and 0 denotes $(2l+1) \times (2l'+1)$ zero matrix.

The random variables Z_{lm}^+, Z_{lm}^- and Z_{lm}^A, Z_{lm}^B can be characterized further in the similar manner as this was done in [2] for the coefficients of expansions of scalar fields on the sphere. The properties stated in Lemma 1 is all that one needs to apply the same reasonings as in [2]. Although the results appear to be completely the same as for the scalar fields, we present them here for completeness of the exposition and because it is really remarkable to get the same characterization of random coefficients within a more general construction than scalar fields.

Lemma 2 *Under the condition of mean square continuity and weak isotropy of the random function $X_+(g)$, the the random variables Z^+_{lm} have the following properties: $ReZ^+_{lm} = ImZ^+_{lm}$, the ratio $\frac{ReZ^+_{lm}}{ImZ^+_{lm}}$ is distributed accordingly to a Cauchy distribution; ReZ^+_{lm} and ImZ^+_{lm} are uncorrelated with variance $\frac{1}{2}$; the marginal distributions of ReZ^+_{lm} and ImZ^+_{lm} are symmetric: $ReZ^+_{lm} = -ReZ^+_{lm}$, $ImZ^+_{lm} = -ImZ^+_{lm}$. The same properties hold for the random variables Z^-_{lm} and Z^A_{lm}, Z^B_{lm} under the conditions of mean square continuity and isotropy of the corresponding fields.*

Lemma 3 *Let the random function $X_+(g)$ be mean square continuous and weakly isotropic. If we assume in addition that $X_+(g)$ is Gaussian, then Z^+_{lm}, $l = 0, 1, ...$, $m = -l, ...l$, are independent Gaussian variables. The same is true for the random variables Z^-_{lm} and Z^A_{lm}, Z^B_{lm} under the corresponding conditions on the underlying fields.*

Note that to prove the independence in the above lemma the statement (2) of Lemma 1 is essential.

Finally, the following very important theorem can be stated.

Theorem 3 *Let the random function $X_+(g)$ be mean square continuous and weakly isotropic. Then for all l, the coefficients $\left\{Z^+_{lm},\ \ m = -l, ..., l\right\}$ are independent if and only if they are Gaussian. The same is true for the random variables Z^-_{lm} and Z^A_{lm}, Z^B_{lm} under the corresponding conditions on the underlying fields.*

The proof is based the Skitovich-Darmois theorem, which gives the criterion for Gaussianity of a collection of independent random variables via independence of some linear statistics of these variables. For detail we refer to [2].

Let us return to the representations (4.35) and (4.36). For the rotation $g = g(\varphi_1, \theta, \varphi_2)$ we know that the inverse rotation g^{-1} is given by the Euler angles $(\pi - \varphi_2, \theta, \pi - \varphi_1)$. Taking into account the expression for T^l_{mn} (see (2.1)) and canceling the factor $e^{\pm 2i\varphi_2}$ in both sides of (4.35) and (4.36) respectively, we come to the representations

$$\begin{aligned} 2\left(X_{\varphi\varphi}(\theta,\varphi) + iX_{\theta\varphi}(\theta,\varphi)\right) &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_l T^l_{2m}(0,\theta,\pi-\varphi_1) Z^A_{lm} \\ &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_l T^l_{2m}(0,\theta,\pi/2+\varphi) Z^A_{lm}, \end{aligned} \tag{4.41}$$

$$\begin{aligned} 2\left(X_{\varphi\varphi}(\theta,\varphi) - iX_{\theta\varphi}(\theta,\varphi)\right) &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} B_l T^l_{-2,m}(0,\theta,\pi-\varphi_1) Z^B_{lm} \\ &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} B_l T^l_{-2,m}(0,\theta,\pi/2+\varphi) Z^B_{lm} \end{aligned} \tag{4.42}$$

(we recall that $\varphi = \pi/2 - \varphi_1$).

We can also reformulate the above results using Wigner D-functions.

The spectral representation for the fields $A(g)$ and $B(g)$ can be obtained as special cases of Proposition 5. More precisely, due to the difference in the convention for rotations, we have to consider

$$\begin{aligned} 2\left(X_{\theta\theta} + iX_{\theta\varphi}\right) e^{-2i\varphi_2} &\equiv A'(g), \\ 2\left(X_{\theta\theta} - iX_{\theta\varphi}\right) e^{2i\varphi_2} &\equiv B'(g). \end{aligned}$$

Due to the special form of $A'(g)$ and $B'(g)$, in the formula (2.9) only the functions $D^l_{m,-2}(g)$ and $D^l_{m,2}(g)$ will participate respectively. To use Proposition 5 we need also to adjust the conditions imposed on $A'(g)$ and $B'(g)$. Namely, we demand the fields $A'(g)$ and $B'(g)$ to be square integrable and strictly isotropic (as described in Proposition 5). Then we can write the decompositions

$$A'(g) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} \sqrt{\frac{2l+1}{8\pi^2}} D^l_{m,-2}(g), \tag{4.43}$$

$$B'(g) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} b_{lm} \sqrt{\frac{2l+1}{8\pi^2}} D^l_{m,2}(g), \tag{4.44}$$

where

$$a_{lm} = \int_G A'(g) \sqrt{\frac{2l+1}{8\pi^2}} \overline{D^l_{m,-2}}(g) dg, \tag{4.45}$$

$$b_{lm} = \int_G B'(g) \sqrt{\frac{2l+1}{8\pi^2}} \overline{D^l_{m,2}}(g) dg, \tag{4.46}$$

the convergence of series is in $L_2(P(d\omega) \times dg)$ and poinwise in $L_2(P(d\omega))$.

We can also deduce from (4.43)-(4.44) the representations

$$2\left(X_{\theta\theta}(\theta,\varphi) + iX_{\theta\varphi}(\theta,\varphi)\right) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} \sqrt{\frac{2l+1}{8\pi^2}} D^l_{m,-2}(\varphi,\theta,0), \tag{4.47}$$

$$2\left(X_{\theta\theta}(\theta,\varphi) - iX_{\theta\varphi}(\theta,\varphi)\right) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} b_{lm} \sqrt{\frac{2l+1}{8\pi^2}} D^l_{m,2}(\varphi,\theta,0). \tag{4.48}$$

Remark 9 We can see that the key step to derive the spectral decomposition for vector and tensor fields on the sphere is to consider the components which transform in a specific way under rotations about the axis e_r. In harmonic analysis on the sphere there exists a special class of functions which are defined relatively their behavior under the rotations of the basis vectors in the plane tangent to the sphere (i.e. rotations around e_r). These are so-called spin-weighted functions. We recall the definition of these functions (see, [10], [23], [28]), however here we use different convention for rotation than the above authors to be consistent with the literature on CMB polarization. First of all, we note that these functions, similarly to the vector- or tensor-valued functions, are defined relatively to some coordinate system. Let at any point of the sphere we have defined three orthogonal vectors: one normal (radial) e_r and two tangential, (e_1, e_2). A function $f(\theta,\varphi)$ defined on the sphere S_2 is said to have spin s if under a right-handed rotations of basis vectors (e_1, e_2) by an angle ψ it transforms as $f'(\theta,\varphi) = e^{-is\psi} f(\theta,\varphi)$. Spin functions can be equivalently defined as evaluation at $\varphi_2 = 0$ of any function in $L_2(SO(3))$ resulting from an expansion for fixed index n in the Wigner D-functions $D^l_{mn}(\varphi_1,\theta,\varphi_2)$. Thus, the functions $D^l_{mn}(\varphi,\theta,0)$ or $D^{l*}_{m,-n}(\varphi,\theta,0)$ define an orthogonal basis for the expansions of spin n functions in $L_2(S_2)$. After normalization in $L_2(S_2)$, these basis functions, which are called the spin-weighted spherical harmonics of spin n, are given in a factorized form in terms of the real Wigner d-functions $d^l_{mn}(\theta)$ and the complex exponentials $e^{im\varphi}$ as follows:

$${}_nY^l_m(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}} d^l_{m,-n}(\cos\theta) e^{im\varphi}. \tag{4.49}$$

We refer for the rigorous mathematical theory for spin-weighted functions to [7], [8].

For example, $X_{\theta\theta}(\theta,\varphi) \pm iX_{\theta\varphi}(\theta,\varphi)$ are spin ± 2 functions and the decompositions (4.47)-(4.48) are decompositions in spin ± 2 spherical harmonics.

Example. In cosmological studies the observable linear polarization field is described in terms of two Stokes' parameters Q and U defined in a given direction with respect to the particular choice of axes in a plane perpendicular to the direction of observation. When reference frame is rotated around the direction of observation (direction of propagation) Q and U transform like the components of a 2-dimensional 2-d rank symmetric trace-free tensor. Therefore, linear polarization can be conveniently described as a tensor-valued field on a sphere

$$\frac{1}{2}\begin{pmatrix} Q & U \\ U & -Q \end{pmatrix}$$

defined with respect to the polar basis vectors (e_θ, e_φ), or, equivalently, via combinations $Q \pm iU$.

Using the above theory, we can expand $Q(\theta,\varphi) \pm iU(\theta,\varphi)$ in the spin ± 2 spherical harmonics. For this we must impose the assumption of strict isotropy, as in Proposition 5, then we can write

$$P(\theta,\varphi) \equiv Q(\theta,\varphi) + iU(\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} {}_2p_{lm}\ {}_2Y_m^l(\theta,\varphi), \tag{4.50}$$

$$P^*(\theta,\varphi) \equiv Q(\theta,\varphi) - iU(\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} {}_{-2}p_{lm}\ {}_{-2}Y_m^l(\theta,\varphi) \tag{4.51}$$

both in $L_2(P(d\omega) \times dg)$ and pointwise in $L_2(P(d\omega))$.

However, as our first approach suggests, the above decompositions are also hold for mean square continuous random fields under the assumption on weak (second-order) isotropy pointwise in $L_2(P(d\omega))$. From the theory presented in this section we have some characteristic properties of random coefficients in the above series (for mean square continuous weakly isotropic random fields). We know, in particular, that the coefficients are uncorrelated, for Gaussian fields the coefficients are Gaussian independent variables, and the coefficients are independent if and only if they are Gaussian. Thus, with the rigorous probabilistic framework provided in the paper, expansions of the form (4.50)-(4.51), which are widely used in the literature on CMB polarization, now become operational.

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